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CMSC 302

Induction and Recursion

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Ch 4 in 6th edition
Ch 5 in 7th edition

induce

verb **cause to happen; encourage**

Synonyms for *induce*

activate	bulldoze	steamroll
breed	cajole	sway
bring about	draw	sweet-talk
cause	effect	wheedle
coax	get	argue into
convince	goose	bring around
engender	impel	draw in
generate	incite	get up
lead to	influence	give rise to
motivate	instigate	prevail upon
persuade	make	sell one on
produce	move	set in motion
promote	occasion	suck in
prompt	press	talk into
urge	procure	twist one's arm
abet	soft-soap	win over
actuate	squeeze	

recur

happen again; repeat in one's mind

Synonyms for *recur*

persist reappear
iterate recrudesce
reiterate repeat
return revert
be remembered be repeated
come again come and go
come back crop up again
haunt thoughts return to mind
run through one's mind

Induction and recursion

are related concepts.

Induction is a proof technique,
recursion is a related programming concept.

Induction proof: assume it is true for n, and show it for n+1 using the assumption.

Recursion: assume you know how to **solve the problem when the input size is *n***, and design a **solution for** the case ***n*+1** using the **solution for *n***.

Note that we could have used numbering as (n-1) and n, instead of n and (n+1)!!!

Topics

- Mathematical induction
- Well ordered property
- Second principle of induction
- Recursive definition
- Recursive algorithms

It may be good at this point to refresh yourself on predicates, predicate logic a.k.a. propositional functions. To do this go to my Lecture 2, handouts 02 Propositional Logic.pdf and start reading from page 59, (you may wish to skip 60-63), and continue by slide 64. Focus on slide 66 ☺

- Important math task is to discover and characterize regular patterns or sequences.
- The main mathematical tool to **prove statements about sequences** is **induction.**
- Induction is also a very important tool in computer science because **most programs are repetition of a sequence of statements**.

- **Example on how induction works:**
- Imagine **climbing an infinitely high ladder**. How do you know whether you will be able to reach an arbitrarily high rung?
- Suppose you make the following two assertions about your climbing:
- **1) I can definitely reach the first rung.**
- **2) Once getting to any rung, I can always climb to the next one up.**
- If both statements are true, then by statement 1 you can get to the first one, and by statement 2, you can get to the second. By statement 2 again, you can get to the third, and fourth, etc.
- Therefore, you can climb as high as you wish. Note: **both of these assertions are necessary** for you to get anywhere on the ladder.
- If only statement 1 is true, you have no guarantee of getting beyond the first rung.
- If only statement 2 is true, you may never be able to get started.

- Assume that the rungs of the ladder are numbered with the positive integers (1,2,3...). Assume a specific property that a number might have now. Instead of "reaching an arbitrarily high rung", we can talk about an **arbitrary positive integer having that property *P*.**
- We will use the **shorthand *P*(*n*) to denote the positive integer n having property *P***. How can we use the ladder-climbing technique to prove that *P*(*n*) is true for all positive *n*?

- The two assertions we need to prove are:
- **1) *P*(1) is true**
- **2) for any positive *k*, if *P*(*k*) is true, then *P*(*k*+1) is true**

- Assertion 1 means we must show the property is true for 1; assertion 2 means that **if any number has property *P* then so does the next number.** If we can prove both of these statements, then *P*(*n*) holds for all positive integers, (which is same as climbing to an arbitrary rung of the ladder).

- The **Principle of Mathematical Induction** can be used as a proof method on statements that have a particular form, and it can be stated as follows:

- A **proof by mathematical induction** that a proposition P(n) is true for every positive integer n **consists of two steps**:

- **BASIS CASE**: Show that the proposition P(1) is true.
- **INDUCTIVE STEP**: Assume that P(k) is true for an arbitrarily chosen positive integer k, and show that under that assumption, P(k+1) must be true.

- From these two steps we conclude (by the principle of mathematical induction) that for all positive integers n, P(n) is true.

- Note: **we don't *prove* that P(k) is true** (except for k = 1). Instead, **we show that *if* P(k) is true, then P(k+1) must also be true.** That's all that is necessary according to the Principle of Mathematical Induction. The assumption that P(k) is true is called the **induction hypothesis.**
  - Understand that P(n) and P(k) are not numbers; they are the propositions that are true or false.

4.1 Mathematical Induction

- A powerful, rigorous technique for proving that a predicate $P(n)$ is true for *every* natural number n, no matter how large.
- Essentially a "domino effect" principle.
- Based on a predicate-logic inference rule:

Antecedents Note: it doesn't matter whether we start with 0 or 1 or 11

$P(0)$ Basis step

$\forall k \geq 0\ (P(k) \rightarrow P(k+1))$ Inductive step

$\therefore \forall n \geq 0\ P(n)$

Consequent *"The First Principle of Mathematical Induction"*

The "Domino Effect"

- **Premise #1:** Domino #1 falls.
- **Premise #2:** For every $n \in \mathbb{N}$, if domino #n falls, then so does domino #n+1.
- **Conclusion:** All of the dominoes fall down!

Note: this works even if there are infinitely many dominoes!

Validity of Induction

- **Proof** that $\forall k \geq 0\ P(k)$ is a valid consequent: Given any $k \geq 0$, the 2[nd] antecedent $\forall n \geq 0\ (P(n) \rightarrow P(n+1))$ trivially implies that $\forall n \geq 0\ (n<k) \rightarrow (P(n) \rightarrow P(n+1))$, *i.e.*, that $(P(0) \rightarrow P(1)) \wedge (P(1) \rightarrow P(2)) \wedge \ldots \wedge (P(k-1) \rightarrow P(k))$. Repeatedly applying the hypothetical syllogism rule to adjacent implications in this list $k-1$ times then gives us $P(0) \rightarrow P(k)$; which together with $P(0)$ (antecedent #1) and *modus ponens* gives us $P(k)$. Thus $\forall k \geq 0\ P(k)$. ■

Modus ponens **is a very common way to make conclusions in classical logic, i.e., it's a rule of inference, and it goes as follows:**

If *A*, then *B*.
***A*.**
Therefore, *B*.

See next slide

Modus ponens or **MP** is an abbreviation of

- ***modus ponendo ponens***

  which is an old Latin saying standing for

  "the way that affirms by affirming"

It is **not** a *logical law*,

it is, rather one of the accepted mechanisms **for the construction of proofs**

Example 1:

Let P be the proposition that the sum of the first n odd numbers is n^2; that is,

$$P(n): 1 + 3 + 5 + \ldots + (2n - 1) = n^2$$

Proof

(The nth odd number is $2n - 1$, and the next odd number is $2n + 1$). Observe that $P(n)$ is true for $n = 1$, that is,

$$P(1): 1 = 1^2$$

Assuming $P(n)$ is true, we add $2n + 1$ to both sides of $P(n)$, obtaining

$$1 + 3 + 5 + \ldots + (2n - 1) + (2n + 1) = n^2 + (2n + 1) = (n + 1)^2$$

which is $P(n + 1)$. That is, $P(n + 1)$ is true whenever $P(n)$ is true. By the principle of mathematical induction, P is true for all n.

A little more formally, same proof goes as follows

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Induction Example 1 – Proof by 1st principle

- Prove that the sum of the first n odd positive integers is n^2. That is, prove:

$$\forall n \geq 1 : \underbrace{\sum_{i=1}^{n} (2i - 1) = n^2}_{P(n)}$$

- Proof by induction.
  - Base case: Let n=1. The sum of the first 1 odd positive integer is 1 which equals 1^2.
- Inductive step: Prove $\forall n \geq 1$: $P(n) \rightarrow P(n+1)$.
  - Let $n \geq 1$, assume $P(n)$, and prove $P(n+1)$.

$$\sum_{i=1}^{n+1} (2i - 1) = \left[\sum_{i=1}^{n} (2i - 1) \right] + (2(n+1) - 1)$$
$$= n^2 + 2n + 1$$
$$= (n+1)^2$$

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What actually has just been shown?

- Base case: P(1)
- If P(k) was true, then P(k+1) is true
  - i.e., P(k) → P(k+1)

- We know it's true for P(1)
- Because of P(k) → P(k+1), if it's true for P(1), then it's true for P(2)
- Because of P(k) → P(k+1), if it's true for P(2), then it's true for P(3)
- Because of P(k) → P(k+1), if it's true for P(3), then it's true for P(4)
- Because of P(k) → P(k+1), if it's true for P(4), then it's true for P(5)
- And onwards to infinity

- Thus, it must be true for all possible values of n

- In other words, we showed that:

$$[\mathrm{P}(1) \wedge \forall k (\mathrm{P}(k) \rightarrow \mathrm{P}(k+1))] \rightarrow \forall n\, \mathrm{P}(n)$$

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Example 2:

- Theorem. $\forall n > 0$, $n < 2^n$. Prove it!

Proof. Let $P(n) = (n < 2^n)$

- Base case: $P(1)=(1<2^1)=(1<2)$ => **T**.

- Inductive step: For n>0, prove $P(n) \rightarrow P(n+1)$.
  - Assuming $n<2^n$, prove $n+1 < 2^{n+1}$.
  - Note $n + 1 < 2^n + 1$ (by inductive hypothesis)
    $< 2^n + 2^n = 2 * 2^n$
    $< 2^{n+1}$
  - So $n + 1 < 2^{n+1}$, and we're done.

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Steps in doing an inductive proof:

1) state the theorem, which is the proposition P(n)
2) prove that P(base case) is true
3) state the inductive hypothesis (substitute *k* for *n*)
4) state what must be proven (substitute *k*+1 for *n*)
5) state that you are beginning your proof of the inductive step, and proceed to manipulate the inductive hypothesis (which we assume is true) to find a link between the inductive hypothesis and the statement to be proven. Always state explicitly where you are invoking the inductive hypothesis.
6) **Always finish your proof with something like: P(*k*+1) is true when P(*k*) is true, and therefore P(*n*) is true for all natural numbers.**

Example 3: Prove that for every positive integer n, the sum of the first *n* positive integers is *n*(*n*+1)/2. This is the *classic* example of an inductive proof.

Note that to begin the inductive step, we state the inductive hypothesis by writing out the meaning of *P*(*k*), then we state what is to be proved based on that hypothesis, *P*(*k*+1). We obtain *P*(*k*+1) by substituting *k*+1 for *k* in *P*(*k*). Writing out *P*(*k*+1) at this point will often show you what is needed in the proof.

Theorem. The following proposition is true for all positive integers $P(n)$: $1+2+3+....+n = n(n+1)/2$

BASE CASE: $P(1)$ asserts that $1 = 1(1+1)/2 = 1$, which is true.

INDUCTIVE STEP:
Assume for some integer k, $P(k)$: $1+2+3+...+k = k(k+1)/2$
Show: $P(k+1)$: $1+2+3+...+(k+1) = (k+1)((k+1)+1)/2$

Proof of the Inductive Step:
By the induction hypothesis, we already have a formula for the first *k* integers.
So, a formula for the first (*k*+1) integers may be found by simply adding (*k*+1) to both sides of the induction hypothesis, and simplifying:

$$\begin{aligned} 1+2+3+...+k+(k+1) &= k(k+1)/2 + (k+1) \\ &= (k(k+1) + 2(k+1)) / 2 \\ &= (k+1)(k+2)/2 \\ &= ((k+1) ((k+1)+1)) / 2 \end{aligned}$$

Thus *P*(*k*+1) is true when *P*(*k*) is true, and **therefore *P*(*n*) is true for all natural numbers**.

Generalizing Induction

- Rule can also be used to prove $\forall n \geq c\ P(n)$ for a given constant $c \in \mathbb{Z}$, where maybe $c \neq 0$.
  - In this circumstance, the **base case** is to prove $P(c)$ rather than $P(0)$, and the **inductive step** is to prove $\forall n \geq c\ (P(n) \rightarrow P(n+1))$.
- Induction can also be used to prove $\forall n \geq c\ P(a_n)$ for any arbitrary series $\{a_n\}$.
- Can reduce these to the form already shown.

Until now we have been using the so-called **weak** induction. There is a variation called **strong** induction. Rather than assuming that *P*(*k*) is true to prove that *P*(*k*+1) is true,

we assume that *P*(*i*) is true for all *i* where
(basis of induction) $\leq i \leq k$.

From this assumption, we prove *P*(*k*+1). It's stronger in the sense that **we are allowed to come to the same conclusion while assuming more**, but the assumption is a natural one based on our understanding of weak induction. In fact, **weak induction and strong induction are logically equivalent**. That is, assuming either one is a valid rule of inference, we can show that the other is.

Strong or Complete Induction:

BASE CASE: Prove *P*(base) is true
INDUCTION: Assume *P*(*base*), *P*(*base+1*)...*P*(*k*) are true, and prove that *P*(*k*+1) is true.

More formal statement is on next slide

Second Principle of Induction

a.k.a. "Strong Induction"

- Characterized by another inference rule:
  $P(0)$
  $\forall n \geq 0$: $\underbrace{(\forall\ 0 \leq i \leq n,\ P(i))}_{P \text{ is true in } all \text{ previous cases}} \rightarrow P(n+1)$
  $\therefore \forall n \geq 0$: $P(n)$
- The only difference between this and the 1st principle is that:
  - the inductive step here makes use of the stronger hypothesis that $P(i)$ is true for *all* **smaller numbers** $i < n+1$, not just for $i = n$.

- Show that every $n > 1$ can be written as a product $\prod p_i = p_1 p_2 \ldots p_s$ of some series of s prime numbers.
  - Let $P(n)$="n has that property"
- **Base case:** n=2, let s=1, p_1=2.
- **Inductive step:** Let $n \geq 2$. Assume $\forall 2 \leq k \leq n$: $P(k)$. Consider $n+1$. If it's prime, let $s=1$, $p_1=n+1$. Else $n+1=ab$, where $1<a\leq n$ and $1<b\leq n$. Then $a=p_1p_2\ldots p_t$ and $b=q_1q_2\ldots q_u$. Then we have that $n+1 = p_1p_2\ldots p_t\, q_1q_2\ldots q_u$, a product of $s=t+u$ primes.

Thus, if *k* + 1 is composite, it can be written as the product of primes, namely, **those primes in the factorization of *a* and those in the factorization of *b*.**

- Prove that every amount of postage of 12 cents or more can be formed using just 4-cent and 5-cent stamps. $P(n)$="n can be…"
- **Base case:** 12=3(4), 13=2(4)+1(5), 14=1(4)+2(5), 15=3(5), so $\forall 12 \leq n \leq 15$, $P(n)$.
- **Inductive step:** Let $n \geq 15$, assume $\forall 12 \leq k \leq n$ $P(k)$. Note $12 \leq n-3 \leq n$, so $P(n-3)$ is valid, and thus add a 4-cent stamp to get postage for $n+1$.

A jigsaw puzzle consists of a number of pieces. Two or more pieces with matched boundaries can be put together to form a "big" piece. To be more precise, we use the term ***block*** to refer to either a single piece or a number of pieces with matched boundaries that are put together to form a "big" piece. Thus, we can simply say that blocks with matched boundaries can be put together to form another block. Finally, when all pieces are put together as one single block, the jigsaw puzzle is solved. Putting 2 blocks together with matched boundaries is called one move. We shall prove (using strong induction) that for a jigsaw puzzle of n pieces, **it will always take n-1 moves to solve the puzzle**.

BASE CASE: P(1) is true--for a puzzle with 1 piece, **it does not take any moves** to solve it.

INDUCTIVE STEP:

Assume P(i) where 1≤ i ≤ k: for a puzzle with i pieces, it takes i-1 moves to solve the puzzle.

Show that for a puzzle with k+1 pieces, it takes k moves to solve the puzzle.

… it continues

Cont.

Proof of the Inductive Step:
Consider the puzzle with k+1 pieces. For the last move that produces the solution to the puzzle, we have two blocks: one with n_1 pieces and the other with n_2 pieces, where $n_1 + n_2 = k + 1$. These two blocks will then be put together to solve the puzzle. According to the induction hypothesis, it took $n_1 - 1$ moves to put together the one block, and $n_2 – 1$ moves to put together the other block.

Including the last move to unite the two blocks, the total number of moves is equal to

$$[(n_1 - 1) + (n_2 - 1)] + 1 = (k + 1) - 1 = k$$

P(k+1) is true when P(i) is true, where i ≤ k, and therefore P(n) is true for any puzzle size.

One more example: A Comparison

Strong induction vs. non-strong induction

- Determine which amounts of postage can be written with 5 and 6 cent stamps
  - **Prove using both versions of induction**
- Answer: any postage ≥ 20

This one can also be phrased closer to previous pages format:

Prove that any postage ≥ 20 can be written with 5 & 6 cent stamps

Proof by weak induction

- Show base case: P(20):
  - 20 = 5 + 5 + 5 + 5
- Inductive hypothesis: Assume P(*k*) is true
- Inductive step: Show that P(*k*+1) is true
  - If P(*k*) uses a 5 cent stamp, replace one stamp with a 6 cent stamp
  - If P(*k*) does not use a 5 cent stamp, it must use only 6 cent stamps
    - Since *k* > 18, there must be four 6 cent stamps
    - Replace these with five 5 cent stamps to obtain *k*+1

Proof by strong induction

- Show base cases: P(20), P(21), P(22), P(23), and P(24)
  - 20 = 5 + 5 + 5 + 5
  - 21 = 5 + 5 + 5 + 6
  - 22 = 5 + 5 + 6 + 6
  - 23 = 5 + 6 + 6 + 6
  - 24 = 6 + 6 + 6 + 6
- Inductive hypothesis: Assume P(20), P(21), …, P(*k*) are all true
- Inductive step: Show that P(*k*+1) is true
  - We will obtain P(*k*+1) by adding a 5 cent stamp to P(*k*+1-5)
  - Since we know P(*k*+1-5) = P(*k*-4) is true, our proof is complete

The Well-Ordering Property

- **The validity of mathematical induction follows from the Well-Ordering Property (WOP),** which is
- a fundamental axiom of number theory.
- WOP states that every nonempty set of non-negative integers has **a least element**.
- This axiom can be used directly in proofs of theorems relating to sets of integers.

I stopped here last time

it continues …

A standard way to use Well Ordering to prove that some property, P(*n*) holds for every nonnegative integer *n*

To prove that "$P(n)$ is true for all $n \in \mathbb{N}$" using the Well Ordering Principle:

- Define the set, C, of *counterexamples* to P being true. Namely, define[a]

$$C ::= \{n \in \mathbb{N} \mid P(n) \text{ is false}\}.$$

- Assume for proof by contradiction that C is nonempty.
- By the Well Ordering Principle, there will be a smallest element, n, in C.
- Reach a contradiction (somehow) —often by showing how to use n to find another member of C that is smaller than n. (This is the open-ended part of the proof task.)
- Conclude that C must be empty, that is, no counterexamples exist. QED

[a] The notation $\{n \mid P(n)\}$ means "the set of all elements n, for which $P(n)$ is true.

The Well-Ordering Property

- *Well-ordering property Axiom* says that:
- Every non-empty set of non-negative integers has a minimal (smallest, least) element.
  - $\forall\, \emptyset \subset S \subseteq \mathbb{N} : \exists m \in S : \forall n \in S : m \leq n$
- and thus, WOP proves that the Induction is valid because
- This implies that $\{n | \neg P(n)\}$ (if non-empty) has a min. element m, but then the assumption that $P(m-1) \rightarrow P((m-1)+1)$ would be contradicted.

(check also page 278 in 6th edition and 314 in 7th one of the book)

or, in another way

- Why is mathematical induction a valid proof technique?
- The reason comes from the well ordering property for the set of positive integers.
- Suppose we know that *P*(1) is true and that the proposition *P(k)* -> *P*(*k* + 1) is true for all positive integers *k*. To show that *P(n)* must be true for all positive integers *n*, **assume** that there is **at least one positive integer for which *P(n)* is false**. Then the set *S* of positive integers for which *P(n)* is false is nonempty.
- Thus, by the well-ordering property, *S* has a least element, which will be denoted by *m*. We know that *m* cannot be 1, because *P*(1) is true. Because *m* is positive and greater than 1 , *m* - 1 is a positive integer. Furthermore, because *m* - 1 is less than m, it is not in S, so *P*(*m* - 1) must be true. Because the conditional statement *P*(*m* - 1) -> *P(m)* is also true, it must be the case that *P(m)* is true. This contradicts the choice of *m*. Hence, *P(n)* must be true for every positive integer *n.*

and, in **the plainest** English the last statements go as:

You assume that

- the set of integers S for which P(*n*) is false is nonempty. By WOP, there would be a smallest positive integer *k* for which P(*k*) is false.
- You then obtain a contradiction, showing that S must be empty.
- The **contradiction** is derived from the fact that for positive integer *j* with $j < k$, P(*j*) must be true due to the way *k* was chosen.

Example 6 *Proof by WOP now*

- Theorem: Every natural number *n* can be written as a product of primes.
- Proof: Let **S** be the set of natural numbers that cannot be written as a product of primes. Then by the WOP, **S** has a **smallest** element, which we will call *n*. *n* must not be a prime, because if it was, it could be written as a product of one prime, itself.
- Thus $n = rs$ for some numbers such that $1 < r, s < n$. Since both *r* and *s* are smaller than *n*, both can be written as products of primes. But that means that *n* is the product of primes, which is a **contradiction**. Thus the set **S** must be empty.
- In other words, **there is no set of natural numbers that cannot be written as product of primes**

Example 7 ***Proof by both Induction and WOP***

- **Theorem**: Every positive integer *n* is either bigger or equal 1, i.e., $n \geq 1$
- **Proof by induction:**
  - Basis step: $P(1)$ holds because $1 \geq 1$
  - Assuming $P(n)$ is true, i.e., $n \geq 1$, add 1 to both side $n+1 \geq 2 > 1$
    - which is $P(n+1)$. That is, $P(n+1)$ is true, whenever $P(n)$ is true.
- **Proof by WOP:**
  - **Suppose there does exist a positive integer less then 1.**
  - By WOP there must exist a least positive integer *a* such that

$$0 < a < 1$$

  - Multiplying both sides by positive integer ***a***

$$0 < a^2 < a$$

  - Therefore, a^2 is a positive integer less than ***a*** which is also less than 1. This **contradicts** ***a***'s property of being the least positive integer less than 1. **Hence, there exists no positive integer less than 1.**

Recursion

4.3 & 4.4: Recursive Definitions

- In induction, we *prove* all members of an infinite set satisfy some predicate *P* by:
  - proving the truth of the predicate for larger members in terms of that of smaller members.

  Induction proves some **CLOSED FORM EXPRESSION**.
- In *recursive definitions*, we similarly *define* a function, a predicate, a set, or a more complex structure over an infinite domain (universe of discourse) by:
  - **defining the function, predicate value, set membership, or structure of larger elements in terms of those of smaller ones.**
- In *structural induction*, we inductively prove properties of recursively-defined objects in a way that parallels the objects' own recursive definitions.

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Recursion

- *Recursion* is the general term for the practice of **defining an object in terms of *itself***
  - or of part of itself
- An inductive proof establishes the truth of $P(n+1)$ *recursively* in terms of $P(n)$.
- There are also recursive *algorithms*, *definitions*, *functions*, *sequences*, *sets*, and other structures.

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What is a meaning of defining in terms of itself?

For example, let $f(x) = x!$

We can define $f(x)$ as

$f(x) = x * f(x-1)$, or one can also use

$f(x+1) = (x+1) * f(x)$

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… more recursion examples

– Find $f(1)$, $f(2)$, $f(3)$, and $f(4)$, where $f(0) = 1$

a) Let $\mathbf{f(n+1) = f(n) + 2}$

- $f(1) = f(0) + 2 = 1 + 2 = 3$
- $f(2) = f(1) + 2 = 3 + 2 = 5$
- $f(3) = f(2) + 2 = 5 + 2 = 7$
- $f(4) = f(3) + 2 = 7 + 2 = 9$

And, the closed form for this series is
$\mathbf{a_n = 2n + 1,\ n = 0, 1, 2, \ldots}$

b) Let $\mathbf{f(n+1) = 3f(n)}$

- $f(1) = 3 * f(0) = 3*1 = 3$
- $f(2) = 3 * f(1) = 3*3 = 9$
- $f(3) = 3 * f(2) = 3*9 = 27$
- $f(4) = 3 * f(3) = 3*27 = 81$

And, the closed form for this series is
$\mathbf{a_n = 3^n,\ n = 0, 1, 2, \ldots}$

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… more recursion examples

– Find $f(1)$, $f(2)$, $f(3)$, and $f(4)$, where $f(0) = 1$

c) Let $f(n+1) = 2^{f(n)}$

- $f(1) = 2^{f(0)} = 2^1 = 2$
- $f(2) = 2^{f(1)} = 2^2 = 4$
- $f(3) = 2^{f(2)} = 2^4 = 16$
- $f(4) = 2^{f(3)} = 2^{16} = 65536$

… and, the closed form for this series is ?

d) Let $f(n+1) = f(n)^2 + f(n) + 1$

- $f(1) = f(0)^2 + f(0) + 1 = 1^2 + 1 + 1 = 3$
- $f(2) = f(1)^2 + f(1) + 1 = 3^2 + 3 + 1 = 13$
- $f(3) = f(2)^2 + f(2) + 1 = 13^2 + 13 + 1 = 183$
- $f(4) = f(3)^2 + f(3) + 1 = 183^2 + 183 + 1 = 33673$

… and, the closed form for this series is ?

Recursively Defined Functions

- Simplest case:
  One way to define a function f: $\mathbb{N} \to S$ (for any set S) or series $a_n = f(n)$ is to:
  - Define $f(0)$ ($f(1), f(2), \ldots, f(n-1)$).
  - For $n>0$, **define $f(n)$ in terms of $f(0),\ldots,f(n-1)$.**
- Ex. Define the series $a_n :\equiv 2^n$ recursively:
  - Let $a_0 :\equiv 1$.
  - For $n > 0$, let $\mathbf{a_n :\equiv 2a_{n-1}}$.

Another Example

- Suppose we define $f(n)$ for all $n \in \mathbb{N}$ recursively by:
  - Let $f(0)=3$
  - For all $n \in \mathbb{N}$, let $f(n+1) = 2f(n)+3$

This is a PURE CLASSIC RECURSION

- What are the values of the following f-s ?
  - $f(1)=$ 9 $f(2)=$ 21 $f(3)=$ 45 $f(4)=$ 93

Recursive Definition of Factorial

- Give an inductive (recursive) definition of the factorial function,
  $F(n) :\equiv n! :\equiv \prod_{1 \le i \le n} i = 1 \cdot 2 \cdot \ldots \cdot n$.
  - Base case: $F(0) :\equiv 1$
  - Recursive part: $F(n) :\equiv n \cdot F(n-1)$.
    - $F(1)=1$
    - $F(2)=2$
    - $F(3)=6$
    - $F(4)=24$ …

More Easy Examples

- Write down recursive definitions for:
  - $a \cdot n$ (a real, n natural) using only addition
  - a^n (a real, n natural) using only multiplication
  - $\sum_{0 \le i \le n} a_i$ (for an arbitrary series of numbers $\{a_i\}$)
  - $\prod_{0 \le i \le n} a_i$ (for an arbitrary series of numbers $\{a_i\}$)

It's over with easy stuff!!! From now on it's getting much tougher.

Fibonacci sequence is a good old, and often useful today, series

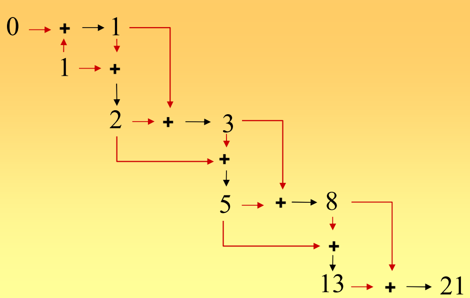
- Definition of the Fibonacci sequence
  - Non-recursive, or closed form: $F(n) = \dfrac{\left(1+\sqrt{5}\right)^n - \left(1-\sqrt{5}\right)^n}{\sqrt{5} \cdot 2^n}$
  - Recursive: $F(n) = F(n\text{-}1) + F(n\text{-}2)$
    or: $F(n\text{+}1) = F(n) + F(n\text{-}1)$
- Must always specify base case(s)!
  - $F(1) = 0$, $F(2) = 1$
  - Note that some will use $F(0) = 0$, $F(1) = 1$

The Fibonacci Series

- The *Fibonacci series* $f_{n \ge 0}$ is a famous series defined by:

  $f_0 :\equiv 0, \quad f_1 :\equiv 1, \quad f_{n \ge 2} :\equiv f_{n-1} + f_{n-2}$

0+1=1, 1+1=2, 1+2=3, 2+3=5, 3+5=8 or simply,
0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, …, infinity.



Leonardo Fibonacci
1170-1250

Inductive Proof about Fib. Series – Upper Bound

Run Fibonacci_Bounds.m now

- Theorem. $f_n < 2^n$.

Proof. By induction. ← Implicitly for all $n \in \mathbb{N}$

Base cases: $f_0 = 0 < 2^0 = 1$, $f_1 = 1 < 2^1 = 2$ } Note use of base cases of recursive def'n.

- Inductive step: Use 2nd principle of induction (strong induction). Assume $\forall k<n,\ f_k < 2^k$.
- Then $f_n = f_{n-1} + f_{n-2}$ and obviously

$$f_n < 2^{n-1} + 2^{n-2} < 2^{n-1} + 2^{n-1} = 2^n. \blacksquare$$

A lower bound on Fibonacci series

- Theorem. For all integers $n \geq 3$, $f_n > \alpha^{n-2}$, where
  $\alpha = (1+5^{1/2})/2 \approx 1.61803$.

We'll show soon where is this number coming from

Proof. (Using strong induction.)

- Let $P(n) = (f_n > \alpha^{n-2})$.
- **Base cases:**
  For n=3, $\alpha^{3-2} = \alpha \approx 1.61803 < 2 = f_3$.
  For n=4, $\alpha^{4-2} = \alpha^2 = (1+2\cdot 5^{1/2}+5)/4 = (3+5^{1/2})/2 \approx 2.61803 < 3 = f_4$.
- **Inductive step:**
  For $k \geq 4$, assume $P(j)$ for $3 \leq j \leq k$, prove $P(k+1)$. Note $\alpha^2 = \alpha+1$.
  $\alpha^{(k+1)-2} = \alpha^{k-1} = \alpha^2\, \alpha^{k-3} = (\alpha+1)\alpha^{k-3} = \alpha^{k-2} + \alpha^{k-3}$. By inductive hypothesis, $f_{k-1} > \alpha^{k-3}$ and $f_k > \alpha^{k-2}$. So,
  $\alpha^{(k+1)-2} = \alpha^{k-2} + \alpha^{k-3} < f_k + f_{k-1} = f_{k+1}$. Thus $P(k+1)$. ■

Vojo, show GOLDEN RATIO SLIDES NOW!!!

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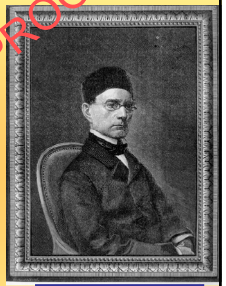
Lamé's Theorem

- Thm. $\forall a,b \in \mathbb{Z}^+$, $a \geq b$, the number of steps in Euclid's algorithm to find gcd(a,b) is $\leq 5k$, where $k = \lfloor \log_{10} b \rfloor + 1$ is the number of decimal digits in b.
  - Thus, Euclid's algorithm is linear-time in the number of digits in b.
- **Proof:**
  - Uses the Fibonacci sequence!
  - See next 2 slides.



Gabriel Lamé
1795-1870

SKIPING LAME THEOREM's PROOF!

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Proof of Lamé's Theorem

- Consider the sequence of division-algorithm equations used in Euclid's algorithm:
  - $r_0 = r_1 q_1 + r_2$ with $0 \leq r_2 < r_1$
  - $r_1 = r_2 q_2 + r_3$ with $0 \leq r_3 < r_2$
  - …
  - $r_{n-2} = r_{n-1} q_{n-1} + r_n$ with $0 \leq r_n < r_{n-1}$
  - $r_{n-1} = r_n q_n + r_{n+1}$ with $r_{n+1} = 0$ (terminate)
- The number of divisions (iterations) is n.

Where $a = r_0$, $b = r_1$, and gcd(a,b)=r_n.

Continued on next slide...

SKIPING LAME THEOREM's PROOF!

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Lamé Proof, continued

- Since $r_0 \geq r_1 > r_2 > \ldots > r_n$, each quotient $q_i \equiv \lfloor r_{i-1}/r_i \rfloor \geq 1$, i=1, ..., n-1.
- Since $r_{n-1} = r_n q_n$ and $r_{n-1} > r_n$, $q_n \geq 2$.
- So we have the following relations between r and f:
  - $r_n \geq 1 = f_2$
  - $r_{n-1} \geq 2r_n \geq 2 = f_3$
  - $r_{n-2} \geq r_{n-1} + r_n \geq f_2 + f_3 = f_4$
  - …
  - $r_2 \geq r_3 + r_4 \geq f_{n-1} + f_{n-2} = f_n$
  - $b = r_1 \geq r_2 + r_3 \geq f_n + f_{n-1} = f_{n+1}$.
- Thus, if n>2 divisions are used, then $b \geq f_{n+1} > \alpha^{n-1}$.
  - Thus, $\log_{10} b > \log_{10}(\alpha^{n-1}) = (n-1)\log_{10}\alpha \approx (n-1)0.208 > (n-1)/5$.
  - If b has k decimal digits, then $\log_{10} b < k$, so $n-1 < 5k$, so $n \leq 5k$.

SKIPING LAME THEOREM's PROOF!

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Recursively Defined Sets

- An infinite set S may be defined recursively, by giving:
  i) A small finite set of *base* elements of S.
  ii) A rule for constructing new elements of S from previously-established elements.
  iii) Implicitly, S has no other elements but these.

- Example. Let $3 \in S$, and let $x+y \in S$ if $x,y \in S$. What is S?

What about this? 3+3=6, 3+6=9, 6+6=12, 3+9=12, …

The Set of All Strings

- Def. Given an alphabet Σ, the set Σ^* of all strings over Σ can be recursively defined by:

  $\lambda \in \Sigma^*$ (λ :≡ "the empty string")

- $w \in \Sigma^* \wedge x \in \Sigma \rightarrow wx \in \Sigma^*$

Note that wx is a string not a product

- Exercise: Prove that this definition is equivalent to our old one: $$\Sigma^* :\equiv \bigcup_{n \in \mathbf{N}} \Sigma^n$$

Other Easy String Examples

- Give recursive definitions for:
- The concatenation of strings $w_1 \cdot w_2$.
- The length $\ell(w)$ of a string w.
- Well-formed formulae of propositional logic involving **T**, **F**, propositional variables, and operators in $\{\neg, \wedge, \vee, \rightarrow, \leftrightarrow\}$.
- Well-formed arithmetic formulae involving variables, numerals, and operations in $\{+, -, *, \uparrow\}$.

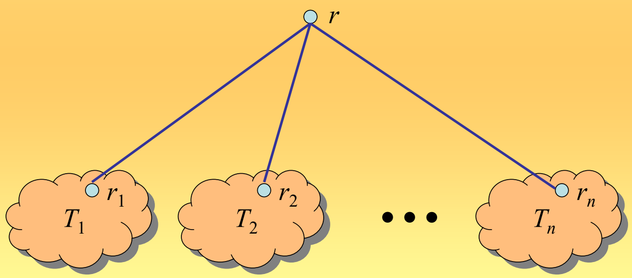
Rooted Trees

- Trees will be covered in more depth in chapter 9.
  - Briefly, a tree is a graph in which there is exactly one undirected path between each pair of nodes.
  - An undirected graph can be represented as a set of unordered pairs (called edges) of objects called nodes.
- Definition of the set of rooted trees:
  i) Any single node r is a rooted tree.
  ii) If $T_1, \ldots, T_n$ are disjoint rooted trees with respective roots $r_1, \ldots, r_n$, and r is a node not in any of the T_i's, then another rooted tree is $\{\{r,r_1\},\ldots,\{r,r_n\}\} \cup T_1 \cup \ldots \cup T_n$.
  iii) That is all.

We will not go into Rooted Trees in this class! Here is the end of Induction and Recursion section

Illustrating Rooted Tree Def'n.

- How rooted trees can be combined to form a new rooted tree…



Draw some examples…

Extended Binary Trees

- A special case of rooted trees.
- Def. Extended binary trees (EBT):

i) The empty set $\varnothing$ is an EBT.

ii) If T_1,T_2 are disjoint EBTs, then $e_1 \cup e_2 \cup T_1 \cup T_2$ is an EBT, where
  $e_1 = \varnothing$ if $T_1 = \varnothing$, and
  $e_1 = \{(r,r_1)\}$ if $T_1 \neq \varnothing$ and has root r_1, and
  similarly for e_2.

iii) That is all.

Draw some examples…

Full Binary Trees

- A special case of extended binary trees.
- Def. Recursive definition of full binary trees (FBT):
  - i) A single node r is a FBT.
    - Note this is different from the EBT base case.
  - ii) If T_1,T_2 are disjoint FBTs, then $e_1 \cup e_2 \cup T_1 \cup T_2$, where
    $e_1 = \varnothing$ if $T_1 = \varnothing$, and
    $e_1 = \{(r,r_1)\}$ if $T_1 \neq \varnothing$ and has root r_1, and
    similarly for e_2.
    - Note this is the same as the EBT recursive case!
      - Can simplify it to "If T_1,T_2 are disjoint FBTs with roots r_1 and r_2, then $\{(r, r_1),(r,r_2)\} \cup T_1 \cup T_2$ is an FBT."
  - i) That is all.

Draw some examples…

Structural Induction

- Proving something about a recursively defined object using an inductive proof whose structure mirrors the object's definition.

Example

- Thm. Let $3\in S$, and let $x+y\in S$ if $x,y\in S$, and that is all. Let $A = \{n\in\mathbb{Z}^+ | (3|n)\}$. Than $A=S$.

  Proof. We show that $A\subseteq S$ and $S\subseteq A$.
  - To show $A\subseteq S$, show $[n\in\mathbb{Z}^+ \wedge (3|n)]\rightarrow n\in S$.
    - **Inductive proof.** Let $P(n) :\equiv n\in S$. Induction over positive multiples of 3. Base case. $n=3$, thus $3\in S$ by def'n. of S. Inductive step. Given $P(n)$, prove $P(n+3)$. By inductive hyp., $n\in S$, and $3\in S$, so by def'n of S, $n+3\in S$.
  - To show $S\subseteq A$: let $n\in S$, show $n\in A$.
    - **Structural inductive proof.** Let $P(n):\equiv n\in A$. Base case. $3\in S$. Since 3|3, $3\in A$. Recursive step. $x,y \in S$, $n=x+y \in S$ and $x,y \in A$. Since $3|x$ and $3|y$, we have $3|(x+y)$, thus $x+y = n \in A$.

- stop

Recursive Algorithms (§4.4)

- Recursive definitions can be used to describe *algorithms* as well as functions and sets.
- Ex. A procedure to compute a^n.

```
procedure power(a≠0: real, n∈ℕ)
    if n = 0 then return 1
else return a · power(a, n−1)
```


Efficiency of Recursive Algorithms

- The time complexity of a recursive algorithm may depend critically on the number of recursive calls it makes.
- Ex. *Modular exponentiation* to a power n can take $\log(n)$ time if done right, but linear time if done slightly differently.
  - Task: Compute b^n **mod** m, where $m\geq 2$, $n\geq 0$, and $1\leq b<m$.

Modular Exponentiation Alg. #1

- Uses the fact that $b^n = b \cdot b^{n-1}$ and that $x \cdot y$ **mod** $m = x \cdot (y$ **mod** $m)$ **mod** m. (Prove the latter theorem at home.)

```
• procedure mpower(b≥1,n≥0,m>b ∈ℕ)
•     {Returns bⁿ mod m.}
  if n=0
      then return 1
      else return (b·mpower(b,n−1,m)) mod m
```

- Note this algorithm takes $\Theta(n)$ steps!

Modular Exponentiation Alg. #2

- Uses the fact that $b^{2k} = b^{k \cdot 2} = (b^k)^2$.

```
• procedure mpower(b,n,m) {same signature}
•     if n=0 then return 1
  else if 2|n
      then return mpower(b,n/2,m)² mod m
      else return (mpower(b,n−1,m)·b) mod m
```

- What is its time complexity? $\Theta(\log n)$ steps

A Slight Variation

- Nearly identical but takes $\Theta(n)$ time instead!

```
• procedure mpower(b,n,m) {same signature}
•     if n=0 then return 1
  else if 2|n
      then return (mpower(b,n/2,m)·
                       mpower(b,n/2,m)) mod m
      else return (mpower(b,n−1,m)·b) mod m
```

The number of recursive calls made is critical!

Recursive Euclid's Algorithm

```
• procedure gcd(a,b∈N)
  if a = 0
      then return b
      else return gcd(b mod a, a)
```

- Note recursive algorithms are often simpler to code than iterative ones…
- However, they can consume more stack space, if your compiler is not smart enough.

Recursive Linear Search

- Note there is no real advantage to using recursion here, rather than just looping **for** *loc* := *i* to *j*…

```
• procedure search(a: series; i, j: integer;
                    x: item to be found)
•      {Find x in series a at a location ≥i and <j}
  if a_i = x
      then return i {At the right item? Return it!}
  if i = j
      then return 0  {No locations in range? Failure!}
  return search(i+1, j, x) {Try rest of range}
```

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Recursive Binary Search

```
• procedure binarySearch(a, x, i, j) {same
  sig}
  {Find location of x in a, ≥i and <j}
  m := ⌊(i + j)/2⌋           {Go to halfway point.}
  if x = a_m
      then return m           {Did we luck out?}
  if x<a_m ∧ i<m         {If it's to the left,}
      then return binarySearch(a,x,i,m−1) {Check
  that ½}
  else if a_m<x ∧ m<j {If it's to right,}
       then return binarySearch(a,x,m+1,j) {Check
  that ½}
  else
      return 0          {No more items, failure.}
```

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Recursive Fibonacci Algorithm

```
• procedure fibonacci(n ∈ N)
  if n=0
      then return 0
  if n=1
      then return 1
  return fibonacci(n−1)+fibonacci(n−2)
```

- Is this an efficient algorithm?
  - Is it polynomial-time in *n*?
- How many additions are performed?

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Analysis of Fibonacci Procedure

- Thm. The preceding procedure for *fibonacci*(n) performs $f_{n+1}-1$ addition operations.

  Proof. By strong structural induction over n, based on the procedure's own recursive definition.
  - **Base cases:** *fibonacci*(0) performs 0 additions, and $f_{0+1}-1 = f_1 - 1 = 1 - 1 = 0$. Likewise, *fibonacci*(1) performs 0 additions, and $f_{1+1}-1 = f_2-1 = 1-1 = 0$.
  - **Inductive step:** For $n>1$, by strong inductive hypothesis, *fibonacci*($n-1$) and *fibonacci*($n-2$) do f_n-1 and $f_{n-1}-1$ additions respectively, and *fibonacci*(n) adds 1 more, for a total of $f_n-1+ f_{n-1}-1+1 = f_n+f_{n-1}+1 = f_{n+1}+1$. ■

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A more efficient algorithm

- **procedure** *findFib*(a,b,m,n)
  {Given $a=f_{m-1}$, $b=f_m$, and $m \leq n$, return f_n}
  if $m=n$
      then return b
  return *findFib*$(b,\ a+b,\ m+1,\ n)$

- **procedure** *fastFib*(n) {Find f_n in $\Theta(n)$ steps.}
  if $n=0$
      then return 0
  return *findFib*$(0,1,1,n)$

Recursive Merge Sort

- **procedure** *sort*$(L = \ell_1, \ldots, \ell_n)$
  if $n>1$
    then {
      $m := \lfloor n/2 \rfloor$ {this is rough ½-way point}
      $L :=$ *merge*(*sort*$(\ell_1, \ldots, \ell_m)$,
          sort$(\ell_{m+1}, \ldots, \ell_n)$)
    }
  return L

- The merge (next slide) takes $\Theta(n)$ steps, and merge-sort takes $\Theta(n \log n)$.

Recursive Merge Method

- **procedure** *merge*$(A,B$: sorted lists)
  {Given two sorted lists $A=(a_1,\ldots,a_{|A|})$, $B=(b_1,\ldots,b_{|B|})$, return a sorted list of all.}
-     **if** $A = ()$
      then return B {If A is empty, it's B.}
  if $B = ()$
      then **return** A {If B is empty, it's A.}
  if $a_1 < b_1$
      then return $(a_1,$ *merge*$((a_2,\ldots a_{|A|}),\ B))$
      else return $(b_1,$ *merge*$(A, (b_2,\ldots,b_{|B|})))$

Merge Routine

- **procedure** *merge*$(A,\ B$: sorted lists)
  L = empty list
  $i:=0,\ j:=0,\ k:=0$
  while $i<|A| \wedge j<|B|$ {$|A|$ is length of A}
    if $i=|A|$ **then** {
      $L_k := B_j$;
      $j := j + 1$
    } **else if** $j=|B|$ **then** {
      $L_k := A_i$;
      $i := i + 1$
    } **else if** $A_i < B_j$ **then {**
      $L_k := A_i$;
      $i := i + 1$
-     } **else** {
-       $L_k := B_j$;
-       $j := j + 1$
-     }
    $k := k+1$
  return L

Takes $\Theta(|A|+|B|)$ time

References

- Rosen
  Discrete Mathematics and its Applications, 6e
  Mc GrawHill, 2007